(2) Transformation of trigonometric functions:

⇒ You should remember that:

- $\sin^2 x = \frac{1}{2}(1-\cos 2x)$, $\cos^2 x = \frac{1}{2}(1+\cos 2x)$, Hence

- $\int \sin^2 x \, dx = \frac{1}{2}\int [1-\cos 2x] dx = \boxed{\frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right] + C}$

- $\int \cos^2 x \, dx = \frac{1}{2}\int (1+\cos 2x) dx = \boxed{\frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right] + C}$

- You know that, $\sin^2 x + \cos^2 x = 1 \longrightarrow (1)$

  → Devide (1) by $\sin^2 x$, Then $\boxed{1 + \cot^2 x = \csc^2 x}$

  → Devide (1) by $\cos^2 x$, Then $\boxed{\tan^2 x + 1 = \sec^2 x}$

- $\int \tan^2 x \, dx = \int (\sec^2 x - 1) dx = \boxed{\tan x - x + C}$

- $\int \cot^2 x \, dx = \int (\csc^2 x - 1) dx = \boxed{-\cot x - x + C}$

⇒ You should remember that:

- $\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$, $\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$.
- $\cosh^2 x - \sinh^2 x = 1 \longrightarrow (2)$

  → Devide (2) by $\cosh^2 x$, Then, $\boxed{1 - \tanh^2 x = \operatorname{sech}^2 x}$

  → // // // $\sinh^2 x$. Then, $\boxed{\coth^2 x - 1 = \operatorname{csch}^2 x}$

- $\int \sinh^2 x \, dx = \frac{1}{2}\int (\cosh 2x - 1) dx = \boxed{\frac{1}{2}\left[\frac{1}{2}\sinh 2x - x\right] + C}$

- $\int \cosh^2 x \, dx = \frac{1}{2}\int (\cosh 2x + 1) dx = \boxed{\frac{1}{2}\left[\frac{1}{2}\sinh 2x + x\right] + C}$

- $\int \tanh^2 x \, dx = \int (1 - \operatorname{sech}^2 x) dx = \boxed{x - \tanh x + C}$

- $\int \coth^2 x \, dx = \int (1 + \operatorname{csch}^2 x) dx = \boxed{x - \coth x + C}$

(3) Powers of trigonometric functions:

Evaluate: (1) $I = \int \sin^n ax \cdot \cos ax \, dx$.

(2) $J = \int \cos^n ax \cdot \sin ax \, dx$.

Put $u =$ (الدالة المرفوعة للأس)

Ans:

(1) Put $u = \sin ax \Rightarrow du = a \cos ax \cdot dx$, Then

$$I = \int u^n \cdot \frac{du}{a} = \frac{1}{a} \int u^n \, du$$

$$= \begin{cases} \dfrac{\ln|u|}{a}, & \text{for } n = -1 \\ \dfrac{u^{n+1}}{a(n+1)}, & \text{for } n \neq -1 \end{cases} = \begin{cases} \dfrac{1}{a} \ln|\sin ax| + C, & \text{for } n = -1 \\ \dfrac{\sin^{(n+1)} ax}{a(n+1)} + C, & \text{for } n \neq -1 \end{cases}$$

(2) Put $u = \cos ax \Rightarrow du = -a \sin ax \, dx$, Then

$$J = \frac{-1}{a} \int u^n \, du = \begin{cases} \dfrac{-\ln|u|}{a} + C, & \text{for } n = -1 \\ \dfrac{-u^{n+1}}{a(n+1)} + C, & \text{for } n \neq -1 \end{cases}$$

$$= \begin{cases} \dfrac{-\ln|\cos ax|}{a} + C, & \text{for } n = -1 \\ \dfrac{-\cos^{(n+1)} ax}{a(n+1)} + C, & \text{for } n \neq -1 \end{cases}$$

لاحظ

نجاح هذه الطريقة يعتمد وجود $\cos ax$ أو $\sin ax$ في ال integrand عشان تروح مع ال dx كجزء من ال du.

Evaluate: $I = \int \sin^3 x \, dx$

تجد في هذه المسألة .. الطريقة السابقة لا تصلح .. لأنه لا يوجد $\cos x$ بجانب ال $\sin^3 x$.

Ans:

$$I = \int \sin^2 x \cdot \sin x \, dx$$

$$= \int [1 - \cos^2 x] \sin x \, dx$$

هذا الأسلوب يصلح عندما يكون الأس فردي لدالة ال $\sin x$ أو ال $\cos x$ وتكون لوحدها فقط.

$$= \int \sin x \, dx - \int \cos^2 x \cdot \sin x \, dx$$

انظر الخلف

Put $u = \cos x \Rightarrow du = -\sin x \, dx$, Then

$$I = -\int du + \int u^2 \, du = -u + \frac{1}{3} u^3 + C$$

$$= \boxed{\frac{1}{3} \cos^3 x - \cos x + C}$$

* for odd power of $\cos x$, it will take the following form

$$\cos^{2n+1} x = \cos^{2n} x \cdot \cos x = [\cos^2 x]^n \cdot \cos x = [1-\sin^2 x]^n \cdot \cos x .$$

let $u = \sin x \Rightarrow du = \cos x \, dx$ Then

$$\int \cos^{2n+1} x \, dx = \int \underbrace{(1-u^2)^n}_{\text{وده تقدر تفكه عادي جدا}} \cdot du$$

جداً...

وشكراً

Evaluate: $I = \int \tan^4 x \, dx$

Ans:

$$I = \int \tan^2 x \left[\sec^2 x - 1\right] dx$$

$$= \int \tan^2 x \cdot \sec^2 x \, dx - \int \tan^2 x \, dx$$

$$= \int \tan^2 x \cdot \sec^2 x \, dx - \int \sec^2 x \, dx + \int dx$$

$$= \boxed{\frac{1}{3}\tan^3 x - \tan x + x + C} \quad \#$$

$\boxed{\text{Put: } \tan^2 x = \sec^2 x - 1}$

تصلح لأي $\int \tan^n x \, dx$ إذا كان الـ n فردي أو زوجي.

Evaluate: $J = \int \tan^3 x \, dx$

Ans: $J = \int \tan^2 x \cdot \tan x \, dx = \int (\sec^2 x - 1) \tan x \, dx$

$$= \int \tan x \cdot \sec^2 x \, dx - \int \tan x \, dx$$

$$= \boxed{\frac{1}{2}\tan^2 x + \ln|\cos x| + C} \quad \#$$

Evaluate: $I = \int \sec^6 x \, dx$

Ans: $I = \int \sec^2 x \cdot \sec^4 x \, dx$, $\sec^2 x = 1 + \tan^2 x$

$$= \int \sec^2 x \left[1 + \tan^2 x\right]^2 dx$$

$$= \int \sec^2 x \left(1 + 2\tan^2 x + \tan^4 x\right) dx$$

$$= \int \sec^2 x \, dx + 2\int \tan^2 x \cdot \sec^2 x \, dx + \int \tan^4 x \sec^2 x \, dx$$

$\boxed{\text{Put: } \sec^2 x = 1 + \tan^2 x}$

تصلح لأي $\int \sec^n x \, dx$ إذا كان الـ n زوجي.

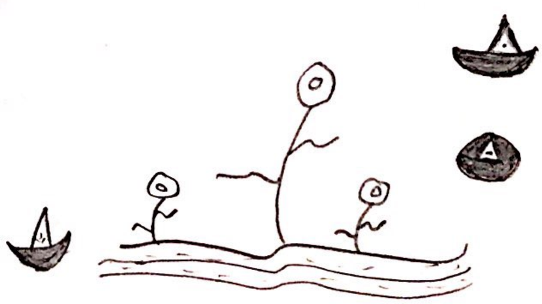
بالمثل: يمكن وضع $\boxed{\csc^2 x = 1 + \cot^2 x}$ عندما يحتوى التكامل $\int \csc^n x \, dx$ وكان n زوجي.

Put $u = \tan x \Rightarrow du = \sec^2 x \, dx$. Then

$$I = \int du + 2\int u^2 \, du + \int u^4 \, du$$

$$= u + \frac{2}{3}u^3 + \frac{1}{5}u^5 + C$$

$$= \boxed{\tan x + \frac{2}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C}$$



(4) Even Powers of sines and Cosines:

* for $\boxed{\int \sin^m x \cdot \cos^n x \, dx}$ →

لو كان عندنا ال m أو ال n -- عدد فردى موجب بنعمل الآتي -- بنشوف صاحب الأس الأكبر وناخد عامل منه عشان يطير مع ال dx كجزء من ال du بعد إزالة الدالة صاحبة الأس الأكبر والذى أصبح زوجى ووضع $\sin^2 x = 1 - \cos^2 x$ إذا كانت دالة ال $\sin x$ ووضع $\cos^2 x = 1 - \sin^2 x$ // // // // $\cos x$ ثم Put $u =$ الدالة اللى بعد الناقص

Ex: Evaluate:

$$I = \int \cos^5 x \cdot \sin^{2/3} x \, dx$$

Ans:

$$I = \int \cos^4 x \cdot \sin^{\frac{2}{3}} x \cos x \cdot dx \quad , \text{ and } \cos^2 x = 1 - \sin^2 x \ , \text{ Then}$$

$$= \int (1 - \sin^2 x)^2 \sin^{\frac{2}{3}} x \cos x \, dx \Rightarrow \text{Put } u = \sin x \Rightarrow du = \cos x \, dx \text{, Then}$$

$$= \int (1 - u^2)^2 u^{\frac{2}{3}} du = \int u^{\frac{2}{3}} (1 - 2u^2 + u^4) \, du$$

$$= \int \left(u^{\frac{2}{3}} - 2u^{\frac{8}{3}} + u^{\frac{14}{3}}\right) du = \frac{3}{5} u^{\frac{5}{3}} - \frac{6}{11} u^{\frac{11}{3}} + \frac{3}{17} u^{\frac{17}{3}} + C$$

$$= \boxed{\frac{3}{5} \sin^{\frac{5}{3}} x - \frac{6}{11} \sin^{\frac{11}{3}} x + \frac{3}{17} \sin^{\frac{17}{3}} x + C}$$

* for $\boxed{\int \sin^m x \cos^n x \, dx}$ → If m, n are even integers, Then use one or both of $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$.

Evaluate: $I = \int \sin^4 x \, dx$.

Ans:

$$I = \int \left[\tfrac{1}{2}(1 - \cos 2x)\right]^2 dx = \frac{1}{4} \int (1 - 2\cos 2x + \cos^2 2x) \, dx$$

$$= \frac{1}{4} \int \left[1 - 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right] dx$$

$$= \frac{1}{4} \int \left(\frac{3}{2} - 2\cos 2x + \frac{1}{2} \cos 4x\right) dx$$

$$= \frac{1}{4} \left[\frac{3}{2} x - \sin 2x + \frac{1}{8} \sin 4x\right) + C$$

$$= \boxed{\frac{3}{8} x - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C}$$

ملحوظة: في حالة وجود $\int \cos^n x \sin^m x \, dx$ وكان كلا من ال m وال n عدد زوجي وكلا الدالتين موجود يتم تغيير أحدهم للآخر بوضع $\sin^2 x = 1 - \cos^2 x$ أو $\cos^2 x = 1 - \sin^2 x$ ونفك القوس ليصبح ال integrand به دالة $\sin x$ أو $\cos x$ فقط ذات أس زوجي يتم التعامل معها كما في المسألة السابقة.

Evaluate: $I = \int \sin^4 x \cos^2 x \, dx$

Ans:

$$I = \int \sin^4 x \cdot (1 - \sin^2 x) \, dx = \int \sin^4 x \, dx - \int \sin^6 x \, dx$$

$$= \int (\sin^2 x)^2 dx - \int (\sin^2 x)^3 dx, \text{ Put } \sin^2 x = \tfrac{1}{2}(1 - \cos 2x), \text{ Then}$$

$$= \overbrace{\frac{1}{4} \int (1 - 2\cos 2x + \cos^2 2x) \, dx}^{K} - \overbrace{\frac{1}{8} \int (1 - \cos 2x)^3 \, dx}^{J}$$

معاملات مفكوك قوس لأس 3

1
1 2 1
1 3 3 1
1 4 6 4 1

دي هتطلع زي المسألة السابقة

$$1 - 3\cos 2x + 3\cos^2 2x - \cos^3 2x$$

$$K = \boxed{\frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x}$$

$$J = \frac{1}{8} \int \left(1 - 3\cos 2x + \frac{3}{2} + \frac{3}{2}\cos 4x\right) dx + \frac{1}{8} \int \cos^2 2x \cdot \cos 2x \, dx$$

$$= \frac{5}{16}x - \frac{3}{16}\sin 2x + \frac{3}{64}\sin 4x - \frac{1}{8} \int \cos 2x (1 - \sin^2 2x) \, dx$$

Put $u = \sin 2x \Rightarrow du = 2\cos 2x \, dx$

$$-\frac{1}{2} \cdot \frac{1}{8} \int (1 - u^2) \, du = \frac{-1}{16} u + \frac{1}{48} u^3$$

$$= \frac{-1}{16}\sin 2x + \frac{1}{48}\sin^3 2x$$

$$I = K - J = \frac{3}{8}x - \cancel{\frac{1}{4}\sin 2x} + \frac{1}{32}\sin 4x$$

$$-\frac{5}{16}x + \cancel{\frac{3}{16}\sin 2x} - \frac{3}{64}\sin 4x + \cancel{\frac{1}{16}\sin 2x} - \frac{1}{48}\sin^3 2x + C$$

$$\therefore \boxed{I = \frac{1}{16}x - \frac{1}{64}\sin 4x - \frac{1}{48}\sin^3 2x + C} \; \#$$

Note that $(x+y)^3 = x^3 + 3x^2 y + 3x y^2 + y^3$;

$(x-y)^3 = x^3 - 3x^2 y + 3x y^2 - y^3$.

Evaluate: $I = \int \sin^4 x \cdot \cos^2 x \, dx$

Another sol^n:

$$I = \int \sin^2 x \cdot \sin^2 x \cdot \cos^2 x \, dx = \int (\sin x \cos x)^2 \sin^2 x \, dx$$

$$= \int \left(\tfrac{1}{2}\sin 2x\right)^2 \cdot \tfrac{1}{2}(1-\cos 2x)\, dx$$

$$= \frac{1}{8}\int \tfrac{1}{2}(1-\cos 4x)(1-\cos 2x)\, dx$$

$$= \frac{1}{16}\int \left[1 - \cos 2x - \cos 4x + \cos 4x \cdot \cos 2x\right] dx$$

$$= \frac{1}{16}\int \left[1 - \cos 2x - \cos 4x + \tfrac{1}{2}\cos 2x + \tfrac{1}{2}\cos 6x\right] dx$$

$$= \boxed{\frac{1}{16}\left[x - \frac{1}{4}\sin 2x - \frac{1}{4}\sin 4x + \frac{1}{12}\sin 6x\right] + C}$$

Note

$\cos\alpha \cdot \cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$

* You should use one of the following formulas to deal with integrals involving the product of sines and cosines of two different angles,

$$\sin\alpha \cdot \sin\beta = \tfrac{1}{2}\left\{\cos(\alpha-\beta) - \cos(\alpha+\beta)\right\}$$

$$\cos\alpha \cos\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$

$$\sin\alpha \cos\beta = \tfrac{1}{2}\left[\sin(\alpha-\beta) + \sin(\alpha+\beta)\right]$$

$$\sin\beta \cos\alpha = \tfrac{1}{2}\left[\sin(\beta-\alpha) + \sin(\beta+\alpha)\right]$$

ابدأ بالزاوية الي مع sin

Evaluate: $J = \int \sin 5x \cos 4x \, dx$

Ans: $J = \int \frac{1}{2}\left[\sin(5x-4x) + \sin(5x+4x)\right] dx$

$$= \frac{1}{2}\int \sin x \, dx + \frac{1}{2}\int \sin 9x \, dx$$

$$= \boxed{-\frac{1}{2}\cos x - \frac{1}{18}\cos 9x + C}$$

Evaluate: $I = \int \frac{\sec x}{\tan^2 x} dx$

Ans: $I = \int \frac{\cos^2 x}{\sin^2 x} \cdot \frac{1}{\cos x} dx = \int \sin^{-2} x \cos x \, dx$ Put $u = \sin x$. $du = \cos x \, dx$

$$= \int u^{-2} du = -u^{-1} + C = \boxed{-\csc x + C}$$